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TAUGHT BY:DR SATISH KUMAR
ASSOCIATE PROFESSOR
DEPARTMENT OF MATHEMATICS
DN PG COLLEGE MEERUT

01 / ADM · CH-06 / H-2051 · Dr Satish Kr Gupta

Graph Theory

Graph

A graph G is a pair of vertices and edges namely (V, E) where $V = \{v_1, v_2, v_3, \ldots\}$ whose elements are called vertices and $E = \{e_1, e_2, e_3, \ldots\}$ whose elements are called edges.

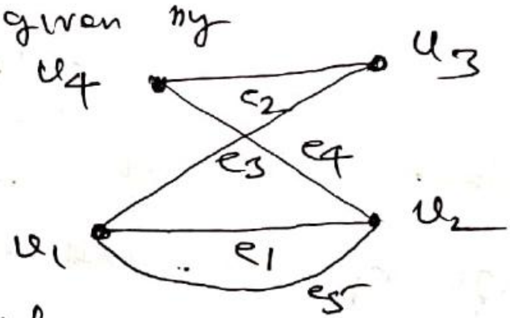
Let $e_k \in E$ then $e_k = (v_i, v_j)$

i.e e_k can be obtained by joining i^{th} vertex of V to j^{th} vertex in V.

Example. Let $V = \{v_1, v_2, v_3, v_4\}$ be the set of vertices and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of edges such that

$e_1 = (v_1, v_2)$, $e_2 = (v_4, v_3)$, $e_3 = (v_1, v_3)$

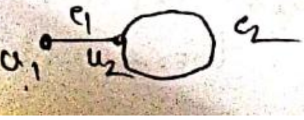
$e_4 = (v_2, v_4)$, $e_5 = (v_1, v_2)$ then the graph $G = (V, E)$ is given by



Parallel edges in a graph

All edges having the same pair of end vertices. In the above figure e_1 and e_5 are parallel edges because they have same pair of end vertices.

Self loop in a graph. It is an edge having same end vertices. eg.



Here, e_2 is a loop

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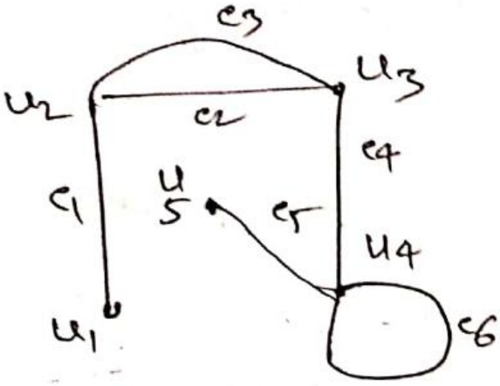
example.

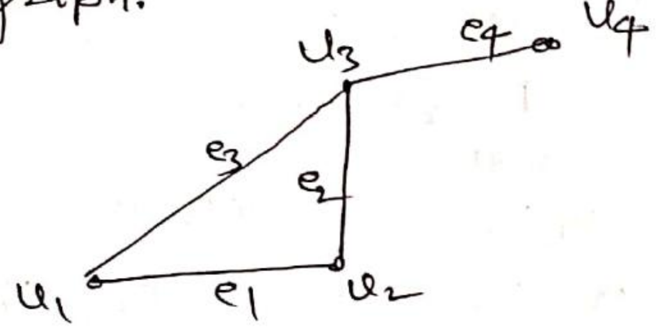


Note: (1) e_2 and e_3 are parallel edges

(2) e_6 is a loop.

Simple graph. A graph that has neither self loops nor parallel edges is called a simple graph.

ex



Simple graph

Incident edge. let $e_k = (v_i, v_j)$ then the edge e_k is said to be incident at v_i and v_j both.

Adjacent vertices. A Two vertices are said to be adjacent vertices if there exist an edge between the given vertices. In the above
v_1 and v_2 are adjacent vertices
v_1 and u_3 are adjacent vertices
but v_1 and u_4 are not adjacent vertices as there is no edge b/w v_1 and u_4.

The degree of a vertex v in a graph G written as $d(v)$, is equal to the number of edges which are incident on v with self loop counted twice.

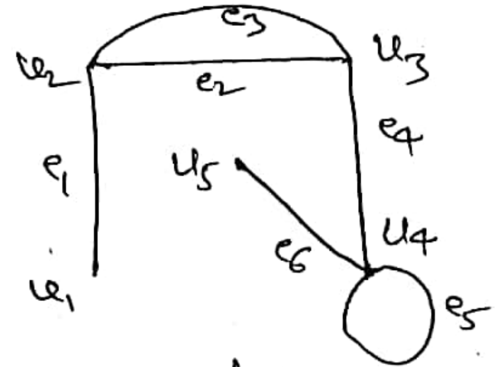
$d(v_1) = 1$

$d(v_2) = 3$

$d(u_3) = 3$

$d(u_4) = 4$

$d(u_5) = 1$



Finite graph A graph $G = (V, E)$ is finite if $O(V)$ = finite i.e no of vertices is finite and $O(E)$ = finite i.e no of edges is finite

Infinite graph. A graph $G = (V, E)$ is called infinite if it is not finite.

Isolated vertex. Any vertex having no edge incident on it is called isolated vertex. Any vertex of degree zero is isolated vertex.

Pendant vertex. A vertex is called Pendant vertex if its degree is one.

Null Graph. A graph with vertices each having zero degree is called Null graph.

ex

Null graph

Theorem. Prove that the sum of the degrees of all the vertices in a graph G is equal to twice the number of edges in G.

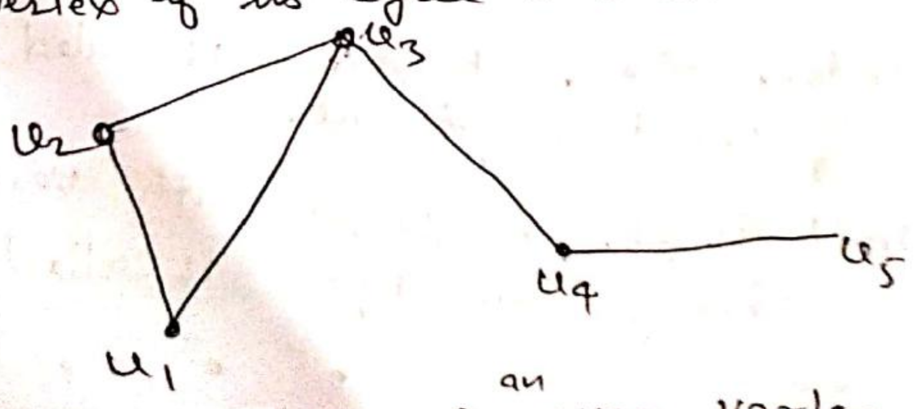
Proof. Let $G=(V,E)$ be a graph where $V = \{u_1, u_2, \dots, u_n\}$. Let e be the no of edges in E. Since each edge is incident on two vertices, it contributes 2 to the sum of degrees of the graph. Thus the sum of degrees of all the vertices in G is twice the number of edges in G.

i.e. $d(u_1) + d(u_2) + \dots + d(u_n) = 2 \times e$

$$\Rightarrow \sum_{i=1}^{n} d(u_i) = 2e$$

even vertex In a graph any vertex is called an even vertex if its degree is even.

odd vertex. In a graph any vertex is called an odd vertex if its degree is odd.



In this graph
u_1 is an even vertex
u_2 is an even vertex
u_4 is an even vertex
u_3 is an odd vertex.

Theorem. Prove that the number of odd vertices in a graph is always even.

Proof. Let $G=(V,E)$ be a graph. Let number of vertices in G is n and no of edges in G is e. then we have

$$\sum_{i=1}^{n} d(v_i) = 2e \qquad \text{---} (1)$$

Let degree of first r vertices is even

Let degree of remaining $(n-r)$ vertices is odd then

(1) can be re-written as

$$\sum_{i=1}^{r} d(v_i) + \sum_{i=r+1}^{n} d(v_i). = 2e$$

$$= \text{An even no}$$

$$\Rightarrow \sum_{i=r+1}^{n} d(v_i) = \text{An even no} - \sum_{i=1}^{r} d(v_i)$$

$$= \text{An even no} - \text{An even no}$$

$$\sum_{i=r+1}^{n} d(v_i) = \text{An even no}$$

$\Rightarrow$ Sum of degree of each odd vertex = even

$\Rightarrow$ number of odd vertices must be even.

proved.

Note. [odd degree + odd degree + --- + rth vertex of odd = even]

$r.v = \text{even}$

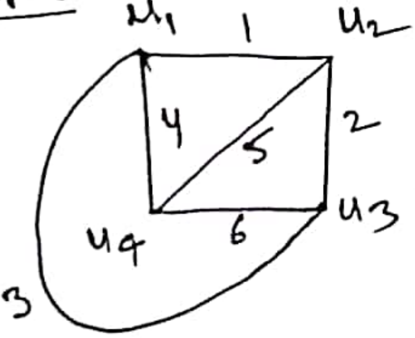
$\Rightarrow r$ must be even

ex
$8 \times 3 = 24$
$16 \times 7 = 112$

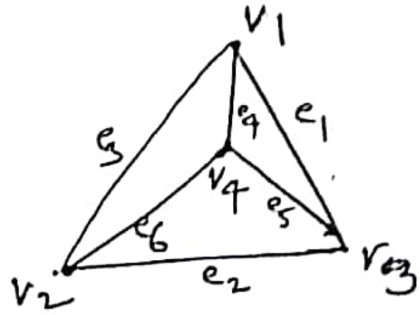
i.e even × odd = even etc

Isomorphic graph. Two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if $\exists$ a one-one correspondence between their vertex set V_1 and V_2 and a one-one correspondence between their edge sets E_1 and E_2 so that the corresponding edges of G_1 and G_2 are incident on the corresponding vertices of G_1 and G_2.

Example.



G_1



G_1

Here $G_1 \cong G_2$

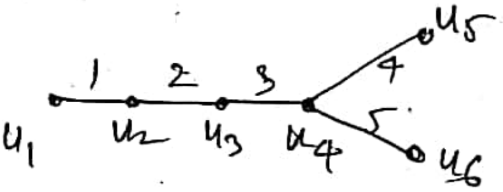
As

$u_1 \longrightarrow v_1$ (u_1 corresponds to v_1)
$u_2 \longrightarrow v_2$ (u_2 corresponds to v_2)
$u_3 \longrightarrow v_3$ (u_3 corresponds to v_3)
$u_4 \longrightarrow v_4$ respectively. (u_4 corresponds to v_4.)

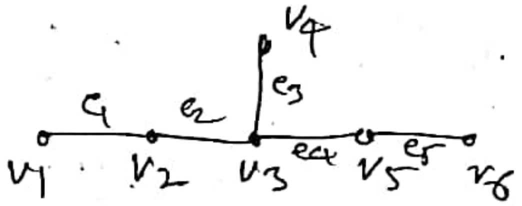
and

$1 \longrightarrow e_1$ (edge 1 corresponds to e_1)
$2 \longrightarrow e_2$ (edge 2 " to e_2)
$3 \longrightarrow e_3$ (edge 3 " to e_3)
$4 \longrightarrow e_4$ (edge 4 " to e_4)
$5 \longrightarrow e_5$ (edge 5 " to e_5)
$6 \longrightarrow e_6$ respectively

Example. (2)



G_1



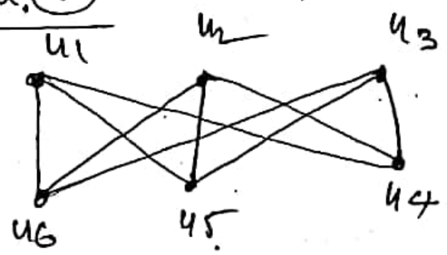
G_2

Here $G_1 \not\cong G_2$ (G_1 is not isomorphic to G_2)

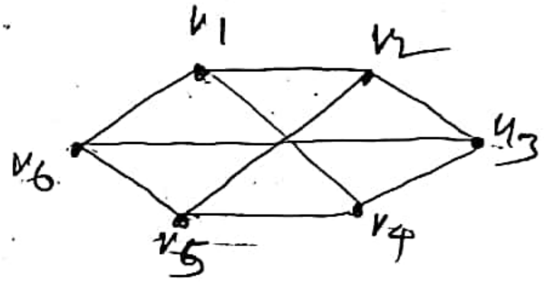
In G_1 degree of u_4 is 3 incident edges at u_4 has degree 2, 1, 1 respectively

In G_2 deg v_3 is 3 incident edges has degree 2, 1, 2 respect.

Example. (3)



G_1

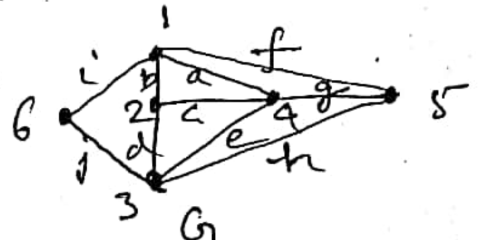


G_2

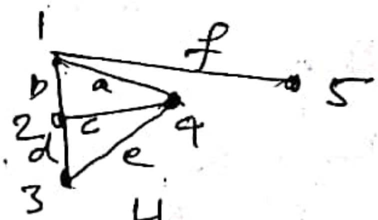
and $G_1 \cong G_2$

Subgraph Let $G = (V, E)$ be a graph. Then a subcollection $H = \{V', E'\}$ of G is said to subgraph of G if H is itself a graph where $V' \subseteq V$ and $E' \subseteq E$.

e.g.



G

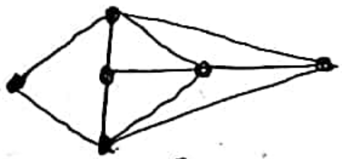


H

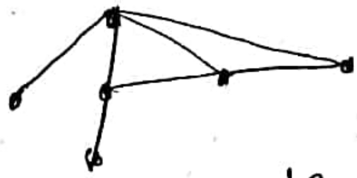
Note. Every graph is its own subgraph.

Spanning subgraph. A subgraph $H = (V', E')$ of a graph $G = (V, E)$ where $V' \subseteq V$, $E' \subseteq E$ is called a spanning subgraph if $V' = V$. i.e if no of vertices in H are equal to the vertices in G.

ex



$G = (V, E)$



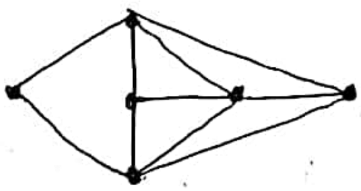
$H = (V, E')$

Here H is a spanning subgraph of G.

Edge-disjoint subgraphs of a graph

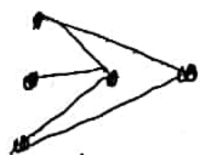
Two subgraphs H_1 and H_2 of a graph G are said to be edge-disjoint if H_1 and H_2 do not have any edge in common.

ex



G

H_1



H_2

Here H_1 and H_2 have no edge in common $\Rightarrow$ H_1 & H_2 are edge disjoint

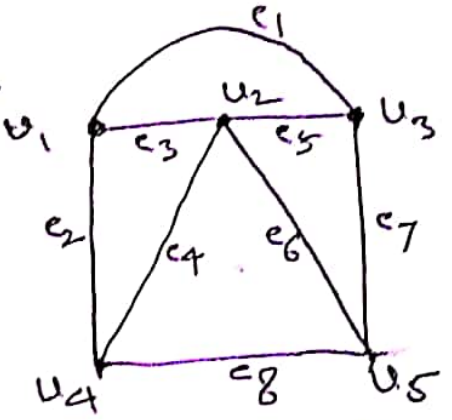
Vertex-disjoint subgroups. Two subgroups H_1 and H_2 of a graph G are said to be vertex disjoint if H_1 and H_2 have no vertex in common.

Note. Every vertex disjoint subgroups are edge disjoint.

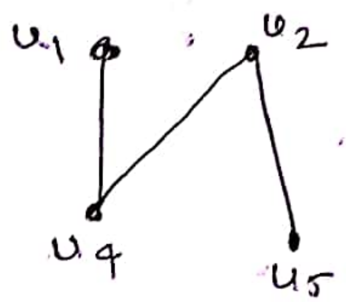
Complement of a Subgraph

Let $H = \{V', E'\}$ of a subgraph of a graph $G = \{V, E\}$ then $\bar{H}(G) = \{V, E - E'\}$ is called complement of H with respect to G.

ex

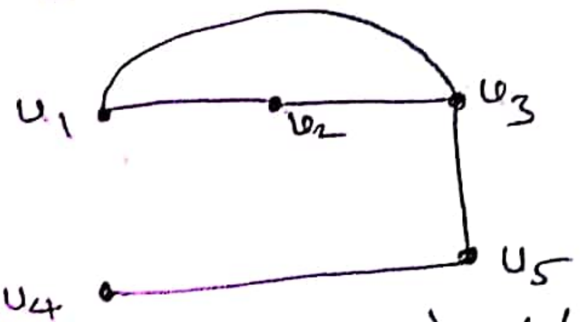


$G = (V, E)$



H

$\bar{H} = \{V, E - E'\}$ is as under

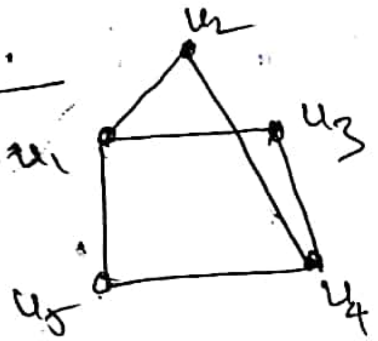


[H की edges G से delete कर देते है बचे हुए graph G को complement of H कहते है]

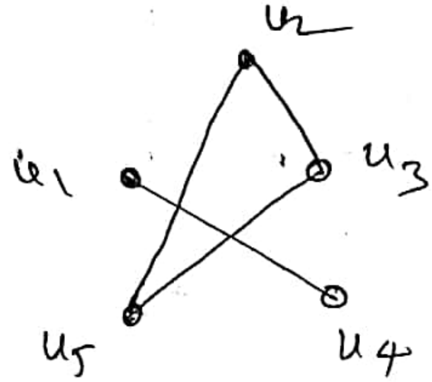
Complement of a simple graph

Let $G = (V, E)$ be a simple graph. then $\bar{G} = (V, E')$ is called complement of G if and only if the edge in E' is not in E.

example.



$G = (V, E)$



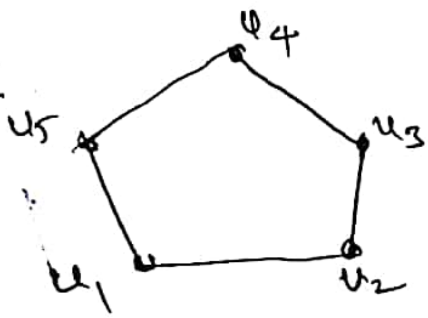
$\bar{G} = (V, E')$

Here $\bar{G}$ is complement of G

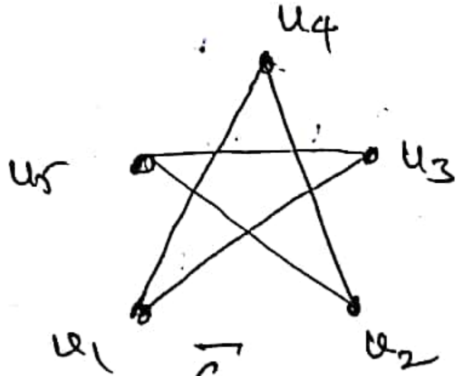
[∵ $\bar{G}$ की edge G में नहीं है]

Self complementary graph . A simple graph which is isomorphic to its complement is called self complementary.

e.g

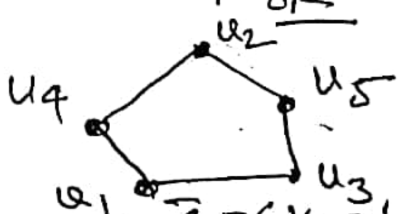


G



$\bar{G}$ OR

i.e



$\bar{G} = (V, E')$

Question. What is the maximum number of vertices in a graph with 35 edges and all vertices are of at least 3.

Soln Let G=(V,E) be a graph. Let n be the number of vertices in G. We know each edge contribute 2 to the sum of degrees of all vertices

$$\Rightarrow \sum d(v_i) = 2e$$
$$\Rightarrow \sum_{i=1}^{n} d(v_i) = 2 \times 35$$
$$= 70$$

It is given each vertex has it degree at least 3. It means

$$3+3+\cdots+3 \le 70$$
(upto n terms)
$$\Rightarrow 3n \le 70$$

Hence maximum no of vertices will be 23.

$$[3 \times 23 \le 70]$$

Question. Prove that if G is self complementary then G has 4k or (4k+1) vertices where k is an integer

Solution. Let $G=(V,E)$ be a self complementary graph having total number of n vertices.
We know that maximum no of edges in a simple graph is $\binom{n}{2}$, Since G is self complementary and isomorphic graphs so they will have equal number of edges namely $\frac{1}{2}\binom{n}{2}$

i.e let $G \cong G'$

Let no of ~~vertices in G~~ edges in $G = \frac{1}{2}\,{}^{n}C_{2}$

then no of edges in $G' = \frac{1}{2}\,{}^{n}C_{2}$

as maximum no of edges in G is ${}^{n}C_{2}$

$\therefore$ G' must have $\frac{n(n-1)}{4}$ edges

$\Rightarrow$ n must be a multiple of 4

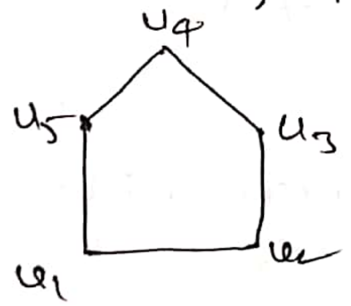
or $(n-1)$ must be a multiple of 4

$\Rightarrow$ $n = 4k$ or $n-1 = 4k$

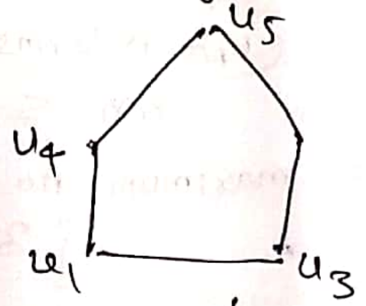
$\Rightarrow n = 4k+1$

, k is an integer.

Note



G



G'

Here G is self complementary

and no of edges in G = no of edges in G'

$\Rightarrow \frac{1}{2}$ (max. no of edges in G) $= \frac{1}{2}$ (max no of edges in G')

Question. Is it possible to have a group of nine people such that each person is acquainted with exactly five of the other people?

Soln. Let us suppose each person as a vertex of a graph $G = (V, E)$, $O(V) = 9$

and each person each person is acquainted with exactly five of the other people it means each person (i.e vertex) has degree 5 so we have

$$\sum_{i=1}^{n} d(v_i) = 2 \times e, \quad e \text{ is no of edges in } G$$

$$\Rightarrow 9+9+9+9+9 = \text{even}$$

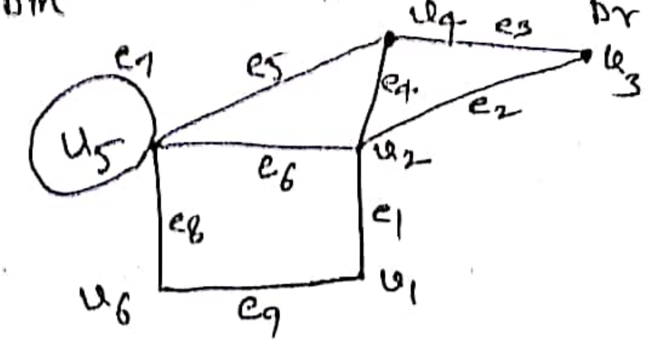
$\rightarrow$ 45 must be an even which is not possible

Hence it is not possible to have a group of 9 people such that each person is acquained with exactly five of the other people.

Walk in a graph

Let $G=(V,E)$ be a graph. A walk in a graph G is a finite alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots v_{k-1}, e_k, v_k$ begining and ending with vertices such that v_{i-1} and v_i are end vertices of the edge e_i, $1 \le i \le k$ and all the edges are distinct.

Note. In a walk any vertex can appears more than once.



Open walk. A walk in a graph is said to be open if its end vertices are distinct.

ex In the above figure $u_1 e_1 u_2 e_2 u_3 e_3 u_4 e_4 u_2 e_6 u_5$ is an open walk.

Closed walk. A walk in a graph is said to be closed if its end vertices are same.

Example. In the above figure $u_2 e_2 u_3 e_3 u_4 e_4 u_2$ is a closed walk.

Path in a graph An open walk in which no vertex appears more than once is called a path.

In the above graph $u_1 e_1 u_2 e_2 u_3 e_3 u_4 e_5 u_5 e_8 u_6$ is a path

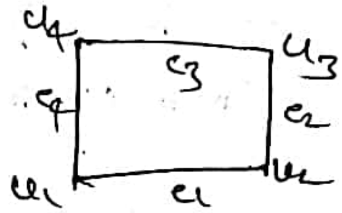
Length of a Path The number of edges in a path is called its length.

ex $u_1 \underline{e_1} u_2 \underline{e_2} u_3 \underline{e_3} u_4 \underline{e_5} u_5 \underline{e_8} u_6$ has 5 length

ie The length of this path is 5

Note. It should be noted in a path degree of end vertices is one and degree of remaining vertices is 2.

Circuit. A closed walk is called a circuit if all its vertices except the end vertices are distinct.



Example. $u_1 e_1 u_2 e_2 u_3 e_3 u_4 e_4 u_1$ is a circuit.

A circuit is also called a cycle.

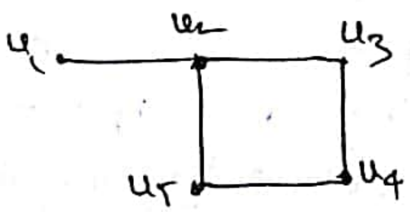
Note. (i) In a circuit every vertex is of degree 2. and the number of edges is equal to the no of vertices

(2) Every self loop in a graph is also a circuit but every circuit is not a self loop.

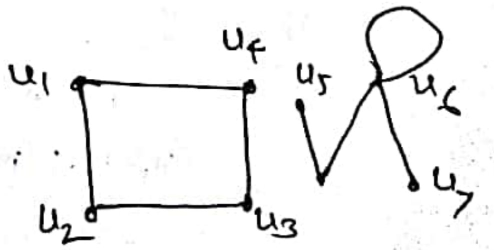
Connected graph A graph $G=(V,E)$ is said to be connected if there exists a path between every pair of vertices in G

Disconnected graph. A graph $G=(V,E)$ which is not connected i.e if $\exists$ no path b/w every pair of vertices in G.

Example



Connected graph
G
having single component.



Disconnected graph having two components of G

Theorem. Prove that a simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof. Let $G=(V,E)$ be a simple graph with n vertices and k components.

Let n_1 be the number of vertices in 1st component of G
n_2 be the number of vertices in 2nd component of G
n_3 be the number of vertices in 3rd component of G
— — — —

n_k be the number of vertices in kth component of G

Then we have

$$n_1+n_2+n_3+\cdots+n_k=n \quad , n_i \geq 1. \quad \text{— (1)}$$

Now, consider the following identity

$$\sum_{i=1}^{k} n_i^2 \leq n^2-(k-1)(2n-k) \quad \text{— (2)}$$

Proof of (2) we have

$$n_1+n_2+n_3+\cdots+n_k=n$$

$$(n_1-1)+(n_2-1)+(n_3-1)+\cdots+(n_k-1)=n-k$$

squaring both sides we have

$$\sum_{i=1}^{k}(n_i-1)^2+\text{Sum of terms of the type } 2(n_i-1)(n_j-1) = n^2+k^2-2nk$$

$$\Sigma(n_i^2-2n_i)+k=n^2+k^2-2nk$$

$$\Rightarrow \Sigma n_i^2-2\Sigma n_i \leq n^2+k^2-2nk-k$$

$$\Rightarrow \Sigma n_i^2 \leq n^2+k^2-2nk-k+2n \quad \left[\sum_{i=1}^{k} n_i=n\right]$$

$$\Rightarrow \sum n_i^2 \le n^2 - [2n(k-1) - k(k-1)]$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k) \quad \text{--- (3)}$$

$$\left[\begin{aligned} &(k-1)(2n-k) \\ &= 2nk - k^2 - 2n + k \\ &= -[k^2 - 2nk + 2n - k] \end{aligned} \right]$$

Since we know that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Therefore, the maximum number of edges in i^{th} component of G is $\frac{n_i(n_i-1)}{2}$

$$\therefore \sum_{i=1}^{k} \frac{1}{2} n_i(n_i-1) = \frac{1}{2}\left[\sum n_i^2 - \sum n_i\right]$$

$$\le \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

$$[\because \sum n_i = n]$$

$$\le \frac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$

$$\le \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$\sum_{i=1}^{k} \frac{1}{2} n_i(n_i-1) \le \frac{1}{2}(n-k)(n-k+1)$$

, proved.

Union of two graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs.

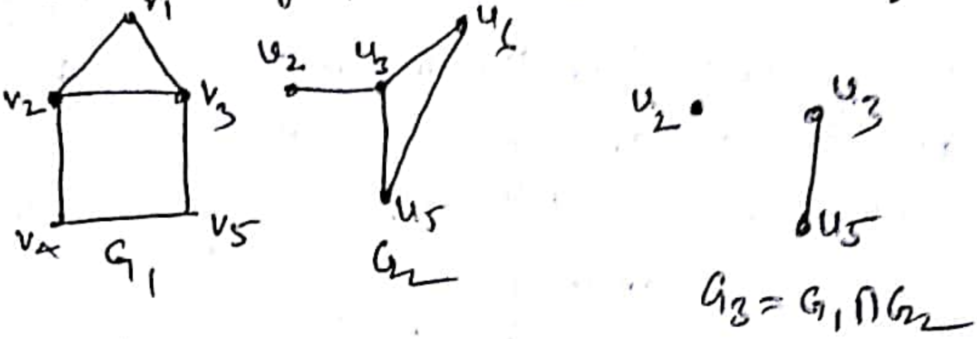
The union of G_1 and G_2 denoted as $G_3 = G_1 \cup G_2$

$= (V_1 \cup V_2, E_1 \cup E_2)$

exp

Intersection of two graphs The intersection of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted as $G_1 \cap G_2$ is the graph $G_3 = (V_1 \cap V_2, E_1 \cap E_2)$

ex



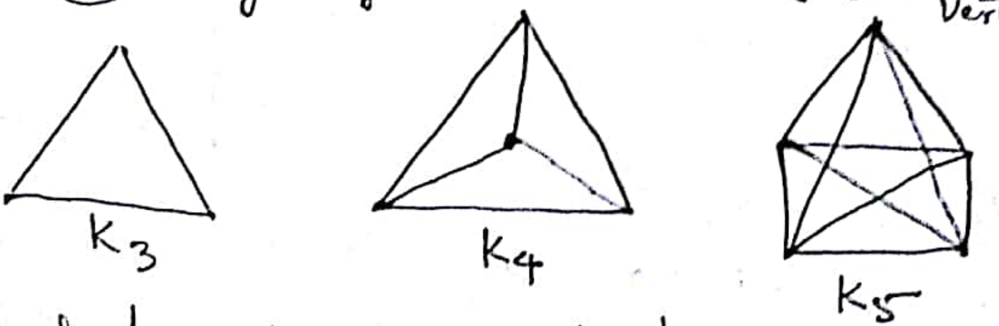
Complete graph

A complete graph G is a simple graph in which each pair of vertices are adjacent. If a complete graph G has n vertices then it will be denoted by K_n.

Note. (1) The no of edges in K_n = ${}^nC_2$

(2) Degree of each vertex = $(n-1)$, n is no of vertices in G.

ex



no of edges in $K_3 = {}^3C_2$

no of edges in $K_4 = {}^4C_2$

no of edges in $K_5 = {}^5C_2$

degree of each vertex in $K_3 = 2$

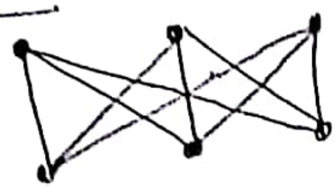
degree of each vertex in $K_4 = 3$

degree of each vertex in $K_5 = 4$

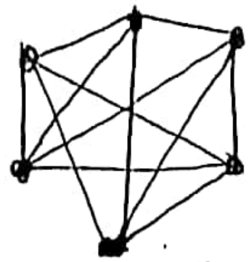
(i.e. no of vertices − 1)

Regular graph A graph G is called regular if each vertex in G has the same degree.
If a graph G is regular with $d(v) = r$ for each vertex v in G then G is called r-regular

Example



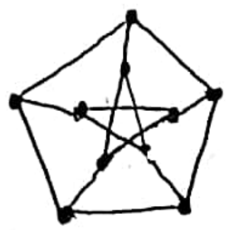
3-regular graph

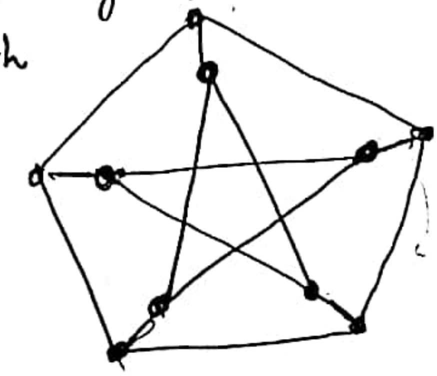


4-regular graph

Petersen graph A 3-regular graph is known as Petersen graph

Example

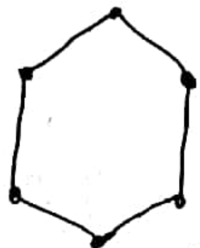




Petersen graph

cycle graph A connected 2-regular graph is called a cycle graph. We denote the cycle graph with n-vertices by C_n

ex



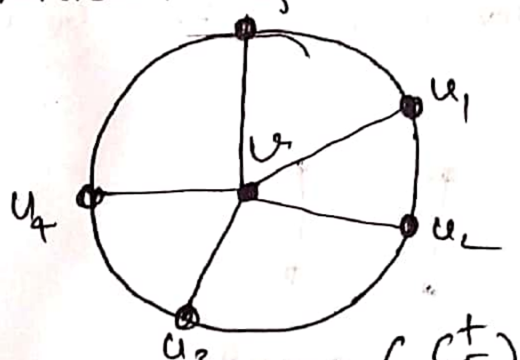
C_6 (cyclic graph)

Path graph It can be obtained by removing an edge from C_n.

P_4

Wheel. The graph obtained from C_{n-1} by joining each vertex to a new vertex v is called a wheel with n vertices.

ex

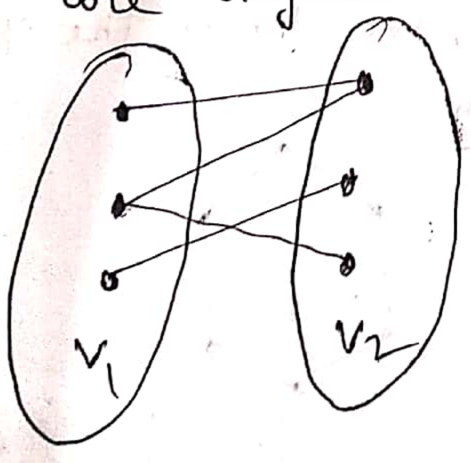


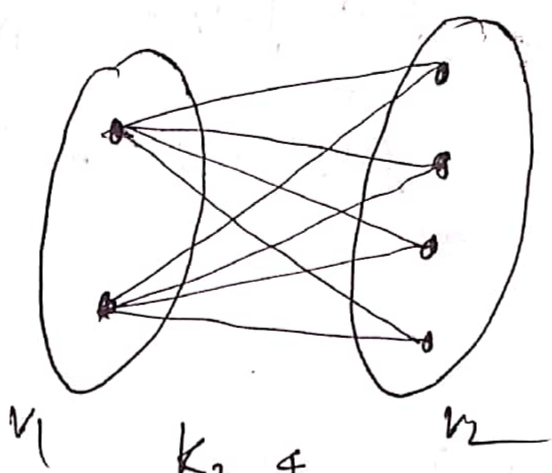
W_6 (C_5^+)

Bipartite graph

A graph $G=(V,E)$ is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joints a vertex in V_1 with a vertex in V_2.

In other words $G=(V,E)$ is bipartite if V can be written as the union of two disjoint sets V_1 and V_2 so that no two members of V_1 or V_2 are adjacent.





$K_{2,4}$

Complete Bipartite graph

Let $G = (V, E)$ be a bipartite graph and let V_1 and V_2 be the partition of the vertex set V of G.
The bipartite graph G is called complete bipartite if each vertex in V_1 is joined to each vertex in V_2 by just one edge. This graph is denoted by $K_{m,n}$ if V_1 has m vertices and V_2 has n vertices.

Note. (1) $K_{m,n}$ has $(m+n)$ vertices
(2) $K_{m,n}$ has mn edges.

(3) G is called k-partite if its vertex set V can be decomposed into k disjoint subsets $V_1, V_2, ---, V_k$ such that each vertex in V_i is joined to each vertex in V_j $(i \neq j)$ by just one edge.

Q. Write Platonic graph

Ans These are the regular graphs namely
(1) tetrahedron
(2) otcahedron
(3) cube
(4) dodecahedron

Question. Show that the maximum number of edges in a complete bipartite graph of n vertices is $\frac{n^2}{4}$.

Soln Let $G = (V, E)$ be a complete bipartite graph such that $V_1 \cup V_2 = V$ and no of vertices in G is n
Let n_1 be the no of vertices in V_1
n_2 be the no of vertices in V_2
then G has $n_1 n_2$ edges where $n_1 + n_2 = n$ — (1)
To show G has maximum no of edges $= \frac{n^2}{4}$

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Let $e = n_1 n_2$

$$e = n_1(n - n_1) \quad \text{——} \quad (1)$$

$$= n_1 n - n_1^2$$

$$\frac{de}{dn_1} = n - 2n_1$$

For maxima and minima

$$\frac{de}{dn_1} = 0 \Rightarrow n - 2n_1 = 0$$

$$\Rightarrow n_1 = \frac{n}{2}$$

$$\Rightarrow n_2 = n_1 - n = \frac{n}{2} - n$$

$$n_2 = \frac{n}{2}$$

$$\therefore e_{max} = n_1 n_2$$

$$= \frac{n}{2} \cdot \frac{n}{2}$$

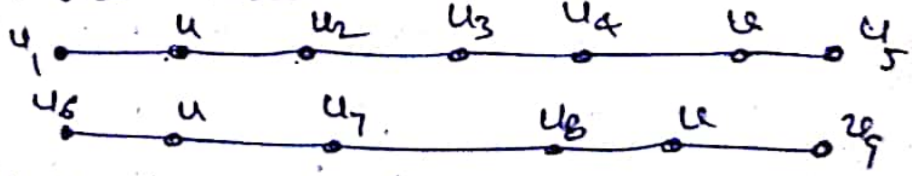
$$= \frac{n^2}{4} \cdot \left(\text{as } \frac{d^2 e}{dn_1} = -2 < 0\right)$$

Hence maximum number of edges in a complete bipartitegraph of n vertices is $\frac{n^2}{4}$.

Question. If the intersection of two paths in a graph is a disconnected graph, show that union of the two paths has atleast one circuit

Soln Let P_1 be

P_2 be



be any two paths such that $(P_1 \cap P_2)$ is disconnected graph. We want to show $(P_1 \cup P_2)$ contains atleast one circuit

Since $P_1 \cap P_2$ is disconnected it means it has a pair of vertices u and v such that there is no edge between u and v in $P_1 \cap P_2$

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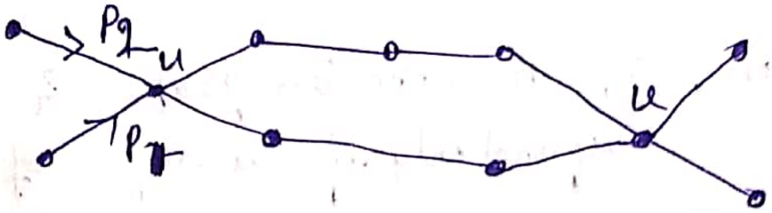


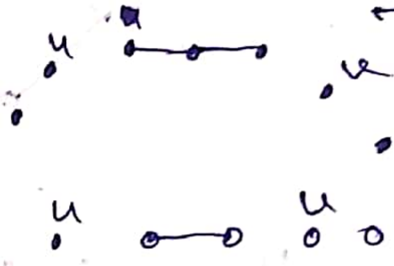
$(P_1 U_2)$

Since u and v are in $(P_1 \cap P_2)$

$\Rightarrow$ both u & v are in P_1 and P_2.

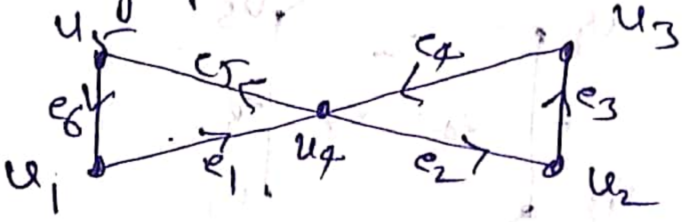
Since P_1 & P_2 are paths $\Rightarrow \exists$ a path in P_1 as well as in P_2 connecting u and v. Since as there is no path in $P_1 \cap P_2$ connecting u & v we may assume that these paths in P_1 and P_2 connecting u & v are edge disjoint. Now union of these two paths connecting u and v forms a circuit in $(P_1 \cup P_2)$ as in fig. above.

Note P_1 से u व v को हटाने पर P_1 becomes
P_2 से u व v को हटाने पर P_2 becomes



Euler graph. A closed walk in a graph which contains all the edges of the graph is called Euler line. A graph which contains Euler line is called Euler graph.

Ex



(no arrow in exam)

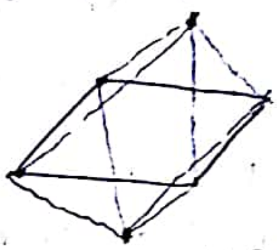
The closed walk $v_1 e_1 v_4 e_2 v_2 e_3 v_3 e_4 v_4 e_5 v_5 e_6 v_1$ is Euler graph as it contains Euler line.

Remember. (1) A connected graph G is an Euler graph if and only if the degree of every vertex is even

(2)

Q Write the name of platonic graph which is Euler graph.

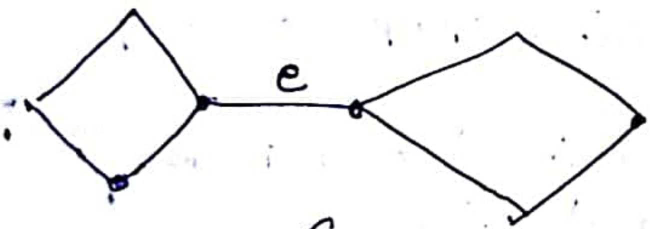
Ans. Octahedron graph.



[जैसे पुराने जमाने में जाली में घी वगैरा रखकर टांग देते थे]

Bridge. An edge in a connected graph G is said to be a bridge if the removal of e from G leaves the graph disconnected

ex

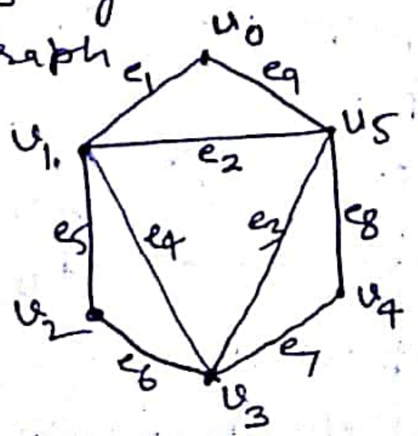


G.

Here $G - \{e\}$ is disconneted

$\Rightarrow$ e is a bridge

Question. Use Fleury's algorithm to construct an Euler line for the graph



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Soln: In the given fig. each vertex is of even degree. It means Euler line will exist. Choose v_0

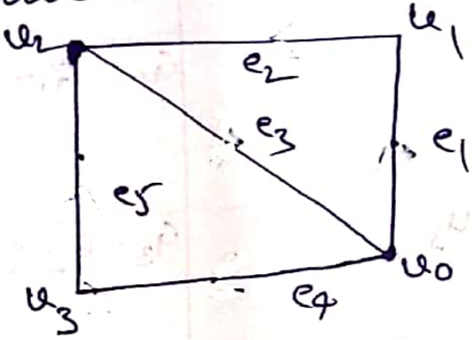
Current walk	G_i	next edge	Reasoning
$T_0: v_0$	$G_0 = G$	e_1	No edge from v_0 is a bridge
$T_1: v_0, e_1, v_1$	$G_1 = G_0 - \{e_1\}$	e_2	No edge from v_1 is a bridge
$T_2: v_0, e_1, v_1, e_2, v_5,$	$G_2 = G_1 - \{e_2\}$	e_3	e_9 is a bridge on G_2 at v_5 choose any one from e_3, e_8
$T_3: v_0, e_1, v_1, e_2, v_5, e_3, v_3$	$G_3 = G_2 - \{e_3\}$	e_4	e_7 is a bridge at v_3 choose e_4 or e_6
$T_4: v_0, e_1, v_1, e_2, v_5, e_3, v_3, e_4, v_1$	$G_4 = G_3 - \{e_4\}$	e_5	only one edge from v_1 is remaining i.e e_5
$T_5: v_0, e_1, v_1, e_2, v_5, e_3, v_3, e_4, v_1, e_5, v_2$	$G_5 = G_4 - \{e_5\}$	e_6	only one edge e_6 is there, so choose e_6
$T_6: v_0, e_1, v_1, e_2, v_5, e_3, v_3, e_4, v_1, e_5, v_2, e_6, v_3$	$G_6 = G_5 - \{e_6\}$	e_7	only one edge e_7 is there, so choose e_7
$T_7: v_0, e_1, v_1, e_2, v_5, e_3, v_3, e_4, v_1, e_5, v_2, e_6, v_3, e_7, v_4$	$G_7 = G_6 - \{e_7\}$	e_8	only one edge e_8 is there so choose it
$T_8: v_0, e_1, v_1, e_2, v_5, e_3, v_3, e_4, v_1, e_5, v_2, e_6, v_3, e_7, v_4, e_8, v_5$	$G_8 = G_7 - \{e_8\}$	e_9	only e_9 is there so choose it
$T_9: v_0, e_1, v_1, e_2, v_5, e_3, v_3, e_4, v_1, e_5, v_2, e_6, v_3, e_7, v_4, e_8, v_5, e_9, v_0$	$G_9 = G_8 - \{e_9\}$		

Thus walk T_9 is the required Euler line.

Hamiltonian circuit A closed walk in a graph G that contains every vertex of G exactly once except the starting vertex.

Hamiltonian graph. It is a graph that has a Hamiltonian circuit.

ex



To: $v_0, e_1, v_1, e_2, v_2, e_5, v_3, e_4, v_0$ is a closed walk that contains every vertex of G exactly once except the starting vertex.